Moment of Inertia

Definition

In everyday speech, the word “moment” refers to a short amount of time. In physics and engineering mechanics, moment is the product of a quantity and the distance from that quantity to a given point or axis. For example, in Statics, a force acting on a wrench handle produces a torque, or moment, about the axis of a bolt: 

\[ M = P \times L \] 

This is the moment of a force.

We can also describe moments of areas. Consider a beam with a rectangular cross-section. The horizontal neutral axis of this beam is the x-x axis in the drawing. Take a small area “a” within the cross-section at a distance “y” from the x-x neutral axis of the beam. The first moment of this area is \( a \times y \). The second moment of this area is \( I_x = (a \times y)^2 = ay^2 \). In Strength of Materials, “second moment of area” is usually abbreviated “moment of inertia”.

If we divide the total area into many little areas, then the moment of inertia of the entire cross-section is the sum of the moments of inertia of all of the little areas.

We can calculate the moment of inertia about the vertical y-y neutral axis: 

\[ I_y = (a \times x)^2 = ax^2 \] 

The “x” and “y” in \( I_x \) and \( I_y \) refer to the neutral axis.

This beam has a depth of 16 cm and a width of 5 cm. We can divide the beam into 8 equal segments 2 cm deep, 5 cm wide, so that each segment has an area \( a = 2 \text{ cm} \times 5 \text{ cm} = 10 \text{ cm}^2 \). The centroid of segment #1 is 7 cm from the x-x axis \( (y_1 = 7 \text{ cm}) \); the centroid of segment #2 is 5 cm from the x-x axis \( (y_2 = 5 \text{ cm}) \); and so on.

We can estimate the moment of inertia for the entire area as the sum of the moments of inertia of the segments, written as

\[ I_x = \sum_{i=1}^{n} a_i y_i^2 \] 

where \( n \) = the total number of segments, and \( i \) = the number of each segment (from 1 to \( n \)), or:

\[ I_x = a_1 y_1^2 + a_2 y_2^2 + a_3 y_3^2 + a_4 y_4^2 + a_5 y_5^2 + a_6 y_6^2 + a_7 y_7^2 + a_8 y_8^2 \]

\[ = 10 \text{ cm}^2 \times (7 \text{ cm})^2 + 10 \text{ cm}^2 \times (5 \text{ cm})^2 + 10 \text{ cm}^2 \times (3 \text{ cm})^2 \]

\[ + 10 \text{ cm}^2 \times (1 \text{ cm})^2 + 10 \text{ cm}^2 \times (1 \text{ cm})^2 + 10 \text{ cm}^2 \times (3 \text{ cm})^2 \]

\[ + 10 \text{ cm}^2 \times (5 \text{ cm})^2 + 10 \text{ cm}^2 \times (7 \text{ cm})^2 \]

\[ = 1680 \text{ cm}^4 \]
We can take the same beam and split it into 16 segments 1 cm deep.

\[ I_s = a_1y_1^2 + a_2y_2^2 + a_3y_3^2 + a_4y_4^2 + a_5y_5^2 + a_6y_6^2 + a_7y_7^2 + a_8y_8^2 \]
\[ + a_9y_9^2 + a_{10}y_{10}^2 + a_{11}y_{11}^2 + a_{12}y_{12}^2 + a_{13}y_{13}^2 + a_{14}y_{14}^2 + a_{15}y_{15}^2 + a_{16}y_{16}^2 \]

We can estimate the moment of inertia as:

\[ I_s = 5 \text{ cm}^2 (7.5 \text{ cm})^2 + 5 \text{ cm}^2 (6.5 \text{ cm})^2 + 5 \text{ cm}^2 (5.5 \text{ cm})^2 \]
\[ + 5 \text{ cm}^2 (4.5 \text{ cm})^2 + 5 \text{ cm}^2 (3.5 \text{ cm})^2 + 5 \text{ cm}^2 (2.5 \text{ cm})^2 \]
\[ + 5 \text{ cm}^2 (1.5 \text{ cm})^2 + 5 \text{ cm}^2 (0.5 \text{ cm})^2 \]
\[ + 5 \text{ cm}^2 (1.5 \text{ cm})^2 + 5 \text{ cm}^2 (2.5 \text{ cm})^2 + 5 \text{ cm}^2 (3.5 \text{ cm})^2 \]
\[ + 5 \text{ cm}^2 (4.5 \text{ cm})^2 + 5 \text{ cm}^2 (5.5 \text{ cm})^2 + 5 \text{ cm}^2 (6.5 \text{ cm})^2 \]
\[ = 1704.525 \text{ cm}^4 \]

As the size of the segments drops, the estimates get closer to the actual solution. If we split the beam into an infinite number of infinitely-small segments, we'll get the actual solution, derived from calculus: \( I_s = \frac{bh^3}{12} \), where \( b \) is the width and \( h \) is the depth.

The exact solution for the moment of inertia of a 5 cm wide, 16 cm deep rectangular beam is

\[ I_s = \frac{bh^3}{12} = \frac{5 \text{ cm} (16 \text{ cm})^3}{12} = 1706.7 \text{ cm}^4. \]

**Compound Beams Sharing a Neutral Axis**

Now take the same beam and glue it side-by-side to an identical beam. The base is twice as wide; the depth is the same. The moment of inertia is \( I_s = \frac{bh^3}{12} = \frac{10 \text{ cm} (16 \text{ cm})^3}{12} = 3413.3 \text{ cm}^4 \), twice the moment of inertia of a single beam. Therefore, we can add the moment of inertia of two cross-sections as long as they share the same neutral axis.

These glued beams share the same neutral axis, so the moment of inertia is

\[ I_s = I_{s1} + I_{s2} = \frac{b_1h_1^3}{12} + \frac{b_2h_2^3}{12} = \frac{b_1h_1^3 + b_2h_2^3}{12} \]
\[ = \frac{5 \text{ cm} (16 \text{ cm})^3 + 3 \text{ cm} (8 \text{ cm})^3}{12} = 1834.7 \text{ cm}^4 \]
This compound beam can be divided into rectangular segments which share a common neutral axis, so:

\[ I_x = I_{x1} + I_{x2} + I_{x3} = \frac{b_1h_1^3 + b_2h_2^3 + b_3h_3^3}{12} \]

**Hollow Beams Sharing a Neutral Axis**

If a beam is hollow, and the hollow space shares the same neutral axis as the beam, then we can subtract the moment of inertia of the hollow from the moment of inertia of an equivalent solid beam.

\[ I_x = I_{x1} - I_{x2} = \frac{b_1h_1^3}{12} - \frac{b_2h_2^3}{12} = \frac{9 \text{ cm}(16 \text{ cm})^3 - 7 \text{ cm}(14 \text{ cm})^3}{12} = 1471 \text{ cm}^4 \]

We can use the same technique for finding the moment of inertia of a hollow tube. From calculus, the moment of inertia of a circle is \( I_x = \frac{\pi d^4}{64} \), therefore the moment of inertia of a hollow circle is

\[ I_x = I_{x1} - I_{x2} = \frac{\pi d_1^4}{64} - \frac{\pi d_2^4}{64} = \frac{\pi (d_1^4 - d_2^4)}{64} \]

A standard 2" steel pipe has dimensions \( d_1 = 2.375 \text{ in.} \) and \( d_2 = 2.067 \text{ in.} \), so:

\[ I_x = \frac{\pi ((2.375 \text{ in.})^4 - (2.067 \text{ in.})^4)}{64} = 0.6657 \text{ in.}^4 \]

**The Transfer Formula**

The moments of inertia calculated in the previous examples were evaluated about the neutral axis of each shape. Sometimes we need to calculate the moment of inertia of a beam about a different, noncentroidal axis. The Transfer Formula is \( I_x = I_{x0} + ad^2 \), where \( I_{x0} = \) moment of inertia about the x-x neutral axis, \( I_x = \) moment of inertia about a parallel x'-x' axis, \( a = \) the area of the shape, and \( d = \) the distance between the x-x neutral axis and the parallel x'-x' axis (the transfer distance). Note: the symbol “d” is also used for the diameter of a circle; these quantities are different, even though they share the same symbol.
For example, this beam has a width of 5 inches and a depth of 6 inches. The moment of inertia about the x-x neutral axis is
\[ I_o = \frac{bh^3}{12} = \frac{5 \text{ in.}(6 \text{ in.})^3}{12} = 90 \text{ in.}^4 \]
and the cross sectional area
\[ a = 5 \text{ in.}\times 6 \text{ in.} = 30 \text{ in.}^2 \]
If we want to find the moment of inertia about the x'-x' axis at the base of the beam, then \( d = 3 \text{ in.} \). Using the Transfer Formula,
\[ I = I_o + ad^2 = 90 \text{ in.}^4 + 30 \text{ in.}^2 (3 \text{ in.})^2 = 360 \text{ in.}^4 \]
The practical application of the Transfer Formula comes with calculating the moment of inertia of compound beams.

**Compound Beams With Different Neutral Axes**

Some compound cross-sections are made of segments which do not share the same neutral axis. As long as the neutral axes are parallel, we can use the Transfer Formula to find the moment of inertia of the compound beam. For example, this beam cross-section consists of two rectangular segments, and the x-x neutral axis of the beam is different from the \( x_1-x_1 \) and \( x_2-x_2 \) neutral axes of segments 1 and 2. Using the Transfer Formula, we can calculate the moment of inertia of each segment about the x-x neutral axis of the compound shape, then add the results to obtain the total moment of inertia.

Using a 10-step process, we can calculate the moment of inertia of the compound beam.

**Step 1** Divide the compound beam into simple shapes, and label the segments. This compound beam can be divided into two segments, but this method also works for complex shapes made up of many simple shapes.

**Step 2** Calculate the area, \( a \), of each segment. Enter the areas and their sum into a table. Be sure to list the units, because in some problems, you may need to include a conversion factor in the calculation.

<table>
<thead>
<tr>
<th>Segment</th>
<th>( a ) (in.(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>24</td>
</tr>
<tr>
<td>#2</td>
<td>48</td>
</tr>
<tr>
<td>Sum</td>
<td>72</td>
</tr>
</tbody>
</table>

\( a_1 = 8 \text{ in.}\times 3 \text{ in.} = 24 \text{ in.}^2 \)
\( a_2 = 4 \text{ in.}\times 12 \text{ in.} = 48 \text{ in.}^2 \)
**Step 3** Pick a Reference Axis, and label it on the diagram. In theory, you can select any axis, but in practice, the math is usually easier if you pick an axis along the top or bottom of the complex shape, or along the centroidal axis of one of the segments.

**Step 4** Draw the distance from the Reference Axis to the centroidal axes of the segments, \( x_1-x_1 \) and \( x_2-x_2 \). Label these distances \( y_1 \), \( y_2 \), etc. Enter these values into the table.

**Step 5** Calculate the product \( a \times y \) for each component area. Enter these values and their sum into the table.

**Step 6** Draw the distance from the Reference Axis to the x-x centroidal axis of the complex shape. Calculate this distance as

\[
\bar{y} = \frac{\sum ay}{\sum a} = \frac{612 \text{ in.}^3}{72 \text{ in.}^2} = 8.5 \text{ in.}
\]
Step 7 Draw the Transfer Distance, d, for each segment. This is the distance from the centroidal axis of the segment to the centroidal axis of the complex shape. Given the way this beam is drawn, $d_1 = y_1 - y$ and $d_2 = y - y_2$. For other compound beams, you will have to figure out the formula for $d_1, d_2, d_3$, etc. based on the drawing.

Enter the results into the table.

<table>
<thead>
<tr>
<th>Segment</th>
<th>a (in.)</th>
<th>y (in.)</th>
<th>ay (in.)</th>
<th>d (in.)</th>
<th>$ad^2$ (in.$^4$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>24</td>
<td>13.5</td>
<td>324</td>
<td>5.0</td>
<td>600</td>
</tr>
<tr>
<td>#2</td>
<td>48</td>
<td>6</td>
<td>288</td>
<td>2.5</td>
<td>300</td>
</tr>
<tr>
<td>Sum</td>
<td>72</td>
<td>612</td>
<td></td>
<td></td>
<td>900</td>
</tr>
</tbody>
</table>

Step 8 Calculate the product $a \times d^2$ for each segment, and enter the results and their sum in the table. Be sure to calculate $a \times d^2$, not $a \times d$...it’s an easy error to make.

Segment 1: $a \times d^2 = 24 \text{ in.}^2 (5 \text{ in.})^2 = 600 \text{ in.}^4$

Segment 2: $a \times d^2 = 48 \text{ in.}^2 (2.5 \text{ in.})^2 = 300 \text{ in.}^4$

Step 9 Calculate I for each segment about its centroidal axis:

$$I_{o1} = \frac{bh^3}{12} = \frac{8 \text{ in.} (3 \text{ in.})^3}{12} = 18 \text{ in.}^4$$

$$I_{o2} = \frac{bh^3}{12} = \frac{4 \text{ in.} (12 \text{ in.})^3}{12} = 576 \text{ in.}^4$$

Enter these values and their sum into the table.

Step 10 Use the Transfer Formula to calculate I for the compound shape.

$$I = \Sigma I_o + \Sigma ad^2 = 594 \text{ in.}^4 + 900 \text{ in.}^4 = 1494 \text{ in.}^4$$

Hollow Beams with Different Neutral Axes

If the beam is hollow and the cavity does not share the same neutral axis as the outline of the solid shape, then we need the Transfer Formula. Let Segment #1 be the solid shape (with no hole), and Segment #2 be the hole. In all calculations, the area of the hole and the moment of inertia of the hole are negative numbers. Thus $a_1, ay_1, ad_1^2$, and $I_1$ are positive numbers; $a_2, ay_2, ad_2^2$, and $I_2$ are negative numbers.
Step 1 Divide the compound beam into simple shapes, and label the segments. Segment #1 is a solid rectangle measuring 8 cm wide by 6 cm deep; Segment #2 is a hole measuring 6 cm wide by 2 cm deep.

Step 2 Calculate the area, \( a \), of each segment.

\[
a_1 = 8 \text{ cm} \times 6 \text{ cm} = 48 \text{ cm}^2
\]

\[
a_2 = -(6 \text{ cm} \times 2 \text{ cm}) = -12 \text{ cm}^2
\]

Step 3 Pick a Reference Axis, and label it on the diagram.

Step 4 Draw the distance from the Reference Axis to the centroidal axes of the segments, \( x_1-x_1 \) and \( x_2-x_2 \). Label these distances \( y_1 \), \( y_2 \), etc. Enter these values into the table.

Step 5 Calculate the product \( a \times y \) for each component area. Enter these values and their sum into the table.
Step 6 Draw the distance from the Reference Axis to the x-x centroidal axis of the complex shape. Calculate this distance as

\[
y = \frac{\sum ay}{\sum a} = \frac{90 \text{ cm}^3}{36 \text{ cm}^2} = 2.5 \text{ cm}
\]

Step 7 Draw the Transfer Distance, d, for each segment. This is the distance from the centroidal axis of the segment to the centroidal axis of the complex shape. Enter the results into the table.

Placing a hole in the upper part of this beam shifts the centroidal axis downward, to where more of the material lies. Since \( d_1 = 0.5 \text{ cm} \), the centroidal axis of this hollow beam is 0.5 cm below the centroidal axis of a solid 6 cm \( \times \) 8 cm solid beam.

### Table

<table>
<thead>
<tr>
<th>Segment</th>
<th>( a ) (cm(^2))</th>
<th>( y ) (cm)</th>
<th>( ay ) (cm(^3))</th>
<th>( d ) (cm)</th>
<th>( ad^2 ) (cm(^4))</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>48</td>
<td>3</td>
<td>144</td>
<td>0.5</td>
<td>12</td>
</tr>
<tr>
<td>#2</td>
<td>12</td>
<td>4.5</td>
<td>54</td>
<td>2</td>
<td>-48</td>
</tr>
<tr>
<td>Sum</td>
<td>36</td>
<td>90</td>
<td>90</td>
<td>-36</td>
<td>140</td>
</tr>
</tbody>
</table>

Step 8 Calculate the product \( a \times d^2 \) for each segment, and enter the results and their sum in the table. Be sure to calculate \( a \times d^2 \), not \( a \times d \)...it’s an easy error to make.

Segment 1: \( a \times d^2 = 48 \text{ cm}^2 (0.5 \text{ cm})^2 = 12 \text{ cm}^4 \)

Segment 2: \( a \times d^2 = -12 \text{ cm}^2 (2 \text{ cm})^2 = -48 \text{ cm}^4 \)

Step 9 Calculate I for each segment about its centroidal axis:

\[
I_1 = \frac{bh^3}{12} = \frac{8 \text{ cm} \times (6 \text{ cm})^3}{12} = 144 \text{ cm}^4
\]

\[
I_2 = -\frac{bh^3}{12} = -\frac{6 \text{ cm} \times (2 \text{ cm})^3}{12} = -4 \text{ cm}^4
\]

Enter these values and their sum into the table.

Step 10 Use the Transfer Formula to calculate I for the compound shape.

\[
I = \sum I_i + \sum ad^2 = 140 \text{ cm}^4 - 36 \text{ cm}^4 = 104 \text{ cm}^4
\]

Some problems have more than two segments; the 10-step procedure is the same, with more rows in the table.
When is the Transfer Formula Not Needed?

Some shapes may look like they require the Transfer Formula, but creative segment choices can make the problem very easy to solve. Consider a beam with two hollow sections. The moment of inertia could be calculated using the 10-step Transfer Formula method with segments #1 (large rectangle), #2 (upper cavity), and #3 (lower cavity), but it is easier to break it into rectangles which share the same centroidal axis.

Calculate the moment of inertia of segment A (large rectangle), subtract the moment of inertia of segment B (two cavities joined together) and add the moment of inertia of segment C (material between the two cavities).

The moment of inertia of a wide-flange beam made of welded rectangular plates is easy to solve by adding and subtracting the moments of inertia of rectangular segments. Subtract the moments of inertia of the spaces to the left and right of the web from the moment of inertia of a large rectangle.

Symbols, Terminology, & Typical Units

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Area</td>
<td>in.², mm²</td>
</tr>
<tr>
<td>B</td>
<td>Transfer distance (in the transfer formula); diameter of a circle</td>
<td>in., mm</td>
</tr>
<tr>
<td>d</td>
<td>Height of a rectangle</td>
<td>in., mm</td>
</tr>
<tr>
<td>I</td>
<td>Moment of inertia</td>
<td>in.⁴, mm⁴</td>
</tr>
<tr>
<td>L</td>
<td>Length</td>
<td>ft., in., m, mm</td>
</tr>
<tr>
<td>M</td>
<td>Moment</td>
<td>ft.lbf., ft.kips, kNm</td>
</tr>
<tr>
<td>P</td>
<td>Point load</td>
<td>lb., kips, N, kN</td>
</tr>
<tr>
<td>y</td>
<td>Distance from the x-x neutral axis to the centroid of a segment</td>
<td></td>
</tr>
</tbody>
</table>