Spatial Analysis 1 – Trigonometry Revisited

By the end of this class you should be able to:
• Determine the coordinates of the tip of a link given its length and angle.
• Determine the length and angle of a link given the coordinates of its tip.

Class Plan
Handouts:
• Basic Trigonometry Review Worksheet (1/student)

Outline
1. Introduction to spatial analysis theme
2. Trig of circle and right triangle: “The one-link robot”
   a) Trig functions – angle to Cartesian
   b) Inverse trig functions – Cartesian to angle

Spatial Analysis Theme

Applications
• Robotics
• Transportation & Facility Planning
• Surveying
• Electromagnetics & Wireless Communication
• Heat Transfer & Fluid Flow
• Molecular Chemistry & Material Science

⇒ Overall much of basic science is spatial

Tools/Skills
• Geometry (plane, solid, ...)
• Trigonometry (now)
• Vector Algebra (next)
• CAD/CAM (in Lab) & Solid Modeling

Previous slide: Spatial Analysis - needs & tools (quick overview)
Our Focus in this course
• We assume basic plane geometry is familiar
  Note the helpfulness of references (handbooks, W. Alpha...)
• We are reviewing Trig quickly ⇒ Focus on a solving triangles approach
• Next we will look at vectors
• We are covering CAD tools in the lab

Have students work through Basic Trig. handout. Then go over next slides

Next Slide: The One-Link Robot
• Used heavily in the book. Simple illustration
• Used to introduce types of problems ⇒ have broader application
• Problem classification is helpful in problem solving
• Note first problem type.
### Basic Definitions

<table>
<thead>
<tr>
<th>Function</th>
<th>Definition for Reference Triangle</th>
<th>Function</th>
<th>Definition for Reference Triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>sine: sin(A)</td>
<td>opposite (= \frac{y}{l})</td>
<td>cosecant: csc(A)</td>
<td>hypotenuse (= \frac{x}{y})</td>
</tr>
<tr>
<td>cosine: cos(A)</td>
<td>adjacent (= \frac{x}{l})</td>
<td>secant: sec(A)</td>
<td>hypotenuse (= \frac{x}{y})</td>
</tr>
<tr>
<td>tangent: tan(A)</td>
<td>opposite (= \frac{\sin(\theta)}{\cos(\theta)})</td>
<td>cotangent: cot(A)</td>
<td>opposite (= \frac{\tan(\theta)}{\cot(\theta)})</td>
</tr>
</tbody>
</table>

### Degrees to Radians

\[ \pi \text{ Radians} = 180 \text{ degrees} \]

\[ \theta_{\text{radians}} = \frac{\pi \text{ radians}}{180^\circ} \theta_{\text{degrees}} \]

### Convert 100º to Radians

\[ \theta_{\text{radians}} = \frac{\pi \text{ radians}}{180^\circ} 100^\circ = 1.75 \text{ radians} \]

### Trigonometric Values

- \(\sin(90^\circ) = 1\), \(\cos(90^\circ) = 0\)
- \(\sin(180^\circ) = 0\), \(\cos(180^\circ) = -1\)
- \(\sin(-90^\circ) = -1\), \(\cos(-90^\circ) = 0\)
- \(\sin(0^\circ) = 0\), \(\cos(0^\circ) = 1\)
- \(\sin(-135^\circ) = -\frac{\sqrt{2}}{2}\), \(\cos(-135^\circ) = -\frac{\sqrt{2}}{2}\)
- \(\sin(135^\circ) = \frac{\sqrt{2}}{2}\), \(\cos(135^\circ) = -\frac{\sqrt{2}}{2}\)
- \(\sin(45^\circ) = \frac{\sqrt{2}}{2}\), \(\cos(45^\circ) = \frac{\sqrt{2}}{2}\)
- \(\sin(-45^\circ) = -\frac{\sqrt{2}}{2}\), \(\cos(-45^\circ) = \frac{\sqrt{2}}{2}\)

Pay Attention to sign pattern.
### Common Reference Angles (Table 3.1)

<table>
<thead>
<tr>
<th>REF ANGLE (RAD)</th>
<th>0°</th>
<th>30°</th>
<th>45°</th>
<th>60°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>sin θ</td>
<td>0</td>
<td>1/2</td>
<td>√3/2</td>
<td>√3</td>
<td>1</td>
</tr>
<tr>
<td>cos θ</td>
<td>1</td>
<td>√3/2</td>
<td>1/2</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>tan θ</td>
<td>0</td>
<td>√3</td>
<td>1</td>
<td>√3</td>
<td>-1</td>
</tr>
</tbody>
</table>

### Example Problem:

Find: \( x \) & \( y \),

Given:
- length \( (l) = 4 \)
- angle \( (\theta) = 120^\circ \)

\[
\sin \theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{y}{l} \\
\Rightarrow y = l \cdot \sin \theta \\
y = 4 \cdot \sin 120 \Rightarrow y = 3.5
\]

or since \( \alpha = 180^\circ - \theta \)

\[
y = l \cdot \sin \alpha \\
y = 4 \cdot \sin 60 \Rightarrow y = 3.5
\]

\[
\cos \theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{x}{l} \\
\Rightarrow x = l \cdot \cos \theta \\
x = 4 \cdot \cos 120 \Rightarrow x = -2
\]

or since \( \alpha = 180^\circ - \theta \)

\[
x = -l \cdot \cos \alpha \\
x = -4 \cdot \cos 60 \Rightarrow x = -2
\]
Work these three problems:

<table>
<thead>
<tr>
<th>Find x &amp; y for links of length ( l ) and the following angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case (1)</td>
</tr>
<tr>
<td>( \theta (°) )</td>
</tr>
</tbody>
</table>

\[ x = cos \theta \rightarrow x = l \cdot cos \theta = 1 \cdot cos(225°) = -0.71 \]
\[ y = sin \theta \rightarrow y = l \cdot sin \theta = 1 \cdot sin(225°) = -0.71 \]

2) \( \theta = 390° \)

\[ x = cos \theta \rightarrow x = l \cdot cos \theta = 1 \cdot cos(390°) = 1.73 \]
\[ y = sin \theta \rightarrow y = l \cdot sin \theta = 1 \cdot sin(390°) = 1 \]

3) \( \theta = -510° \)

\[ x = cos \theta \rightarrow x = l \cdot cos \theta = 1 \cdot cos(-510°) = -2.6 \]
\[ y = sin \theta \rightarrow y = l \cdot sin \theta = 1 \cdot sin(-510°) = -1.5 \]
Given: \( P(x, y) \)

Find \( l = ? \) \quad Find \( \theta = ? \)

\( \text{Pythagorean Theorem: } a^2 + b^2 = c^2 \)

Or for our context \( x^2 + y^2 = l^2 \)

\[ l = \sqrt{x^2 + y^2} \]

\[ \tan \theta = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{y}{x} \]

\[ \Rightarrow \theta = \tan^{-1} \left( \frac{y}{x} \right) = \arctan \left( \frac{y}{x} \right) \]

Need to keep track of quadrant \( \Rightarrow \)

Inverse functions have limited range—highlight table in Basic Trig Worksheet

Calculate \( \tan^{-1}(y/x) \) for:

- **Case 1:** \((x, y) = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)\)
- **Case 2:** \((x, y) = \left( -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)\)

What do you get?

- **Case 1:**
  \[ l = \sqrt{\left( \frac{1}{\sqrt{2}} \right)^2 + \left( \frac{1}{\sqrt{2}} \right)^2} = 1 \]
  \[ \theta = \tan^{-1} \left( \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} \right) = 45^\circ \]
  \[ \text{calculator: } \theta = 45^\circ \]

- **Case 2:**
  \[ l = \sqrt{\left( -\frac{1}{\sqrt{2}} \right)^2 + \left( -\frac{1}{\sqrt{2}} \right)^2} = 1 \]
  \[ \theta = \tan^{-1} \left( \frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} \right) = 0^\circ \]
  \[ \text{calculator: } \theta = 0^\circ \text{ ... wrong!} \]

- Caught by limited range of \( \tan^{-1} \)
  \( \pm 90^\circ \) \quad \( \pm \pi/2 \) radians \quad quadrants I & IV
  quadrants where \( x \) (denominator) is positive
- Signs can tell correct quadrant
- If \( x \) (denominator) is negative \( \Rightarrow \) add \( 180^\circ \) or \( \pi \) radians for quadrants II & III
- Excel has function to keep track \( \text{atan2}(x, y) \), also check your calculator.
Problem
What is the length and angle of a link whose end is at a point (-0.5 ft, 0.25 ft) from its pivot point?

\[ l = \sqrt{x^2 + y^2} = \sqrt{(-0.5)^2 + (0.25)^2} = 0.559 \text{ ft.} \]

For angle
1. Note angle is in quadrant 2
2. Calc. angle with the inv. tan
   \[ \tan^{-1}(y/x) = \tan^{-1}(0.25/-0.50) = -26.57^\circ \text{ (-0.46 radians)} \]
3. The denominator is negative, therefore add 180°.
   The correct angle is: \( 180^\circ - 26.57^\circ = 153.43^\circ \)
   \( (\pi - 0.46 \text{ radians}) = 2.68 \) \( \text{ quadrant } \checkmark \)
   or use atan2 (Excel)

For the given information find the angle of the laser

- What is to be found?
- What is given?
- How would you solve?
**Additional Problem:** If time allows

- **What is to be found?** \( \theta \)
- **What is given?**
  - adjacent side = 5 m
  - opposite side = 2 m
- **How would you solve?** SOHCAHTOA

\[
\frac{\text{opposite}}{\text{adjacent}} = \tan(\theta)
\]

\[
\theta = \tan^{-1}\left(\frac{\text{opposite}}{\text{adjacent}}\right) = \tan^{-1}\left(\frac{2}{5}\right)
\]

\[
\theta = \tan^{-1}(0.4) = 0.38 \text{ radians}
\]

or 21.8°

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**Range of Inverse Trigonometric Functions**

- \( \cos^{-1}(x) \) \( \Rightarrow \) \( 0 \rightarrow \pi \)
- \( \tan^{-1}(x/y) \) \( \Rightarrow \) \( -\frac{\pi}{2} \rightarrow \frac{\pi}{2} \)
- \( \sin^{-1}(y) \) \( \Rightarrow \) \( -\frac{\pi}{2} \rightarrow \frac{\pi}{2} \)
- \( \text{atan2}(x, y) \) \( \Rightarrow \) \( -\pi \rightarrow \pi \)

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**Function Table**

<table>
<thead>
<tr>
<th>function</th>
<th>EXCEL</th>
<th>limited range</th>
</tr>
</thead>
<tbody>
<tr>
<td>arccosine</td>
<td>acos(( \theta ))</td>
<td>0 ( \Rightarrow ) ( \pi )</td>
</tr>
<tr>
<td>arcsine, arctan</td>
<td>asin(( \theta )), ( \text{atan}(\theta) )</td>
<td>(-\pi/2 \Rightarrow \pi/2 )</td>
</tr>
<tr>
<td>Special arctan fn.</td>
<td>( \text{atan2}(x,y) )</td>
<td>(-\pi \Rightarrow \pi )</td>
</tr>
</tbody>
</table>